JOURNAL OF FUNCTIONAL ANALYSIS 63, 240–260 (1985)

Hardy–Littlewood Theory for Semigroups

N. TH. VAROPOULOS*

Université de Paris VI, 4, Place Jussieu, Paris 75230, France

Communicated by Paul Malliavin

Received November 1984

0. INTRODUCTION

0.1. *The Definition of the Dimension*

Let (X, ξ) be a measure space (let us say σ-finite) and let T_t $(t>0)$ be a submarkovian (strongly continuous) symmetric semigroup, i.e., for all $t>0$, T_t: $L^2(X)\to L^2(X)$ is a symmetric operator, and for all $f\in L^2$ with $0\leqslant f\leqslant 1$ we have $0\leqslant T_t f\leqslant 1$.

A vast theory of such semigroups exists (e.g., [1–3, 8]). In particular it is well known that $T_t=e^{-tA}$, where A is a positive self-adjoint operator on L^2. (In [1, 4, 5] the letter A is used for what now is our $-A$.) Let us denote by $D_A\subset L^2$ the domain of A and by $V=D_{A^{1/2}}$ the domain of $A^{1/2}$. For every $f\in V$ let

$$Q(f,f)=(Af,f)=\lim_{t\to 0}\left(\frac{I-T_t}{t}f,f\right). \tag{0.1}$$

(Observe that for every $f\in L^2$ the expression under the above limit is a decreasing function of t. V is exactly the space of f's for which the above limit is finite; cf. [1, Chap. V].) Q is the Dirichlet form attached to the semigroup (cf. [1, 2]).

In the work of Beurling and Deny [1, 2] a locally compact (countable at ∞) topology is usually given to the space X and ξ is a Radon measure. For our purposes, however, this will not be essential.

We shall denote by

$$\mathscr{K}=\{f\in L^\infty(X);\text{ measure }[f\neq 0]<+\infty\}.$$

In the Beurling–Deny theory $\mathscr{K}$ is taken to be

$$\{f\in C(X);\ \operatorname{supp} f\text{ is compact}\}. \tag{0.2}$$

0022-1236/85 $3.00

We shall impose the following regularity conditions on the Dirichlet form Q (of (0.1)):

(i) $\mathscr{K} \cap V$ is dense in V for the norm topology of V given by

$$\|f\|_V = \|f\|_{L^2} + Q^{1/2}(f,f); \qquad f \in V.$$

(ii) $\mathscr{K} \cap V$ is dense in $\mathscr{K}$ for every L^p-topology $(1 \leqslant p < +\infty)$.

Conditions (i) and (ii) are slight modifications of the standard regularity condition of the Beurling–Deny theory (cf. [1, Chap. IV), where instead of (ii) it is required that $\mathscr{K} \cap V$ is dense in $\mathscr{K}$ for the uniform topology (and where $\mathscr{K}$ is as in (0.2)). One cannot go very far in that theory without some kind of "regularity." For our purposes we use these conditions only marginally. As the reader will easily check we can get away with far less.

DEFINITION. Let $(T_t, t>0)$ be a semigroup as above and let Q be the Dirichlet form that it induces (as in (0.1)). We shall then say that the dimension of (T_t) is $n \geqslant 2$, and denote this by

$$\dim(T_t) = n, \tag{0.3}$$

if there exists $C_1 > 0$ s.t.

$$\|f\|_{2n/(n-2)} \leqslant C_1 Q^{1/2}(f,f), \qquad \forall f \in \mathscr{K} \cap V. \tag{0.4}$$

Observe that the dimension of the semigroup T_t could be n *and* m for two different $n, m \geqslant 2$. [This, e.g., is always the case when $(X, \xi) = (\mathbb{N}$, counting measure) (i.e., $\mathbb{N} = \{1, 2, ...\}$ and $\xi(j) = 1$ $(j \geqslant 1)$) for then the spaces $L^p(X)$ are nested (i.e., $l^p \subset l^q$; $1 \leqslant p < q$).]

We shall never the less, abusively, denote the validity of (0.4) by (0.3).

The above definition is motivated by the classical heat diffusion semigroup $P_t =$ (convolution by $c_d t^{-d/2} \exp(-\|x\|^2/4t)$) on euclidean space ($\mathbb{R}^d$, Leb.) with $d \geqslant 3$. In that case we have $\dim(P_t) = d$ (and in fact $\dim(P_t) = d' \Rightarrow d' = d$).

The pivot of this paper is the following:

THEOREM 1. *Let $(T_t; t>0)$ be a symmetric submarkovian semigroup as above and let $n > 2$. Then the following are equivalent:*

(A) $\dim(T_t) = n$.

(B) *There exists $C_2 > 0$ s.t.*

$$\|T_t f\|_\infty \leqslant C_2 t^{-n/2} \|f\|_1, \qquad \forall f \in L^1(X), \qquad t > 0.$$

In fact $(A) \Rightarrow (B)$ even in the case $n = 2$.

The above theorem was motivated by problems in the theory of Markov chains and Riemannian Geometry (cf. [4–7]). We shall refer the reader to these papers for several concrete illustrations. The above theorem is infact essentially proved in [5]. In Section 1 and 2 of this paper we shall show how one can adapt the proof of [5] to our (general) setting.

0.2. *The Hardy–Littlewood Estimates*

In this paper we shall show how the estimate (B) of Theorem 1 suffices to develop most of the classical Hardy–Littlewood–Sobolev theory in the setting of a general semigroup.

DEFINITION. Let $u(t, x)$ $(t>0, x\in X)$ be a function on $(0, +\infty)\times X$. We shall say that u is a harmonic function (with respect to the semigroup $T_t(t>0)$) if

$$\begin{aligned} &u(t,\cdot)\in L^1+L^\infty(X); \qquad t>0\\ &T_t u(s,\cdot)=u(t+s,\cdot). \end{aligned} \tag{0.5}$$

We shall say that u is a subharmonic function if it satisfies (0.5) and the weeker condition

$$T_t u(s,\cdot)\geqslant u(t+s,\cdot). \tag{0.6}$$

Let $u(t, x)$ be a subharminic function and $0<\alpha\leqslant+\infty$ and let as denote

$$m_\alpha(t)=\left\{\int |u(t,x)|^\alpha\,dx\right\}^{1/\alpha}; M_\alpha(u)=\sup_t m_\alpha(t). \tag{0.7}$$

We shall say that a harmonic function $u(t, x)$ belongs to the class H_α $(0<\alpha<+\infty)$ if

$$\sup_t |u(t,x)|=u^*(x)\in L^\alpha(X).$$

For $u\in H_\alpha$, we shall denote by $\|u\|_\alpha=\|u^*\|_{L^\alpha}$ (which is a norm only if $1\leqslant\alpha\leqslant+\infty$). We have then

THEOREM 2. *Let $(T_t; t>0)$ be a submarkovian symmetric semigroup and let $C_2, n>0$ and $1\leqslant p<+\infty$ be such that*

$$\|T_t f\|_\infty\leqslant C_2 t^{-n/2p}\,\|f\|_p; \qquad t>0,\quad f\in L^p. \tag{0.8}$$

Then for every subharmonic function $u(t, x)$ and every $0<\alpha<\beta\leqslant+\infty$ we have

$$m_\beta(t)\leqslant Ct^{-(n/2)((1/\alpha)-(1/\beta))}M_\alpha(u),$$

where C only depends on C_2, n, p, α, and β.

For every harmonic function $u(t, x)$ we shall define

$$G_z u(t, \cdot) = \int_0^\infty s^{z/2-1} u(t+s), \cdot)\, ds; \qquad (z \in \mathbb{C}) \tag{0.9}$$

(principal value of $s^{z/2}$) provided that the integral converges absolutely. G_z can in fact be thought as an operator given formally by

$$G_z = \int_0^\infty s^{z/2-1} T_s\, ds \tag{0.10}$$

(observe that if $z = \alpha$ is real then the formal expression (0.10) always makes sense since $0 \leqslant G_\alpha f \leqslant +\infty$ for all $0 \leqslant f \in \mathscr{K}$). By elementary spectral theory on $L^2(X)$ we also have

$$G_z = k(z)\, A^{-z/2}, \qquad (\operatorname{Re} z > 0), \tag{0.11}$$

where $k(z)$ is a constant depending on z. In the classical case of the heat diffusion semigroup on $\mathbb{R}^d$ ($d \geqslant 3$), G_α is the standard Riesz potential given by the convolution kernel $G_\alpha(x, y) = k(\alpha, d)\, \|x-y\|^{\alpha-d}$ $(0 < \alpha < d)$. We have then

THEOREM 3. *Let us assume that the semigroup $(T_t; t > 0)$ satisfies the condition* (0.8) *of Theorem* 2 *for some fixed C_2, n and p. Then for every $0 < \alpha < \beta$ and $z \in \mathbb{C}$ such that $1/\beta = (1/\alpha) - (\operatorname{Re} z/n)$ and any $u \in H_\alpha$ the integral* (0.9) *converges absolutely and $G_z u \in H_\beta$.*

For $\alpha > 1$ an alternative proof of the above is contained in [23].

(0.3) *Holomorphic Semigroups and the Time Derivative*

It is well known that for every $1 < q < +\infty$, a submarkovian symmetric semigroup $(T_t; t > 0)$ is automatically holomorphic on $L^q(X)$ (cf. [3, 8]). This implies that for every $f \in \mathscr{K}$ and $u(t, x) = T_t f(x)$ we have

$$\frac{\partial^k}{\partial t^k} u(t, \cdot) \in L^q, \qquad 1 < q < +\infty,$$

where the derivatives $\partial/\partial t$ are taken in the Banach space sense (the Banach space being L^q). Furthermore for $k = 1, 2, \ldots,$ the norms

$$\sup_{t>0} t^k \left\| \frac{\partial^k}{\partial t^k} u(t, \cdot) \right\|_q = \|f\|_q^{(k)}; \qquad f \in \mathscr{K}$$

are all equivalent norms on $\mathscr{K}$ and we have

$$\|f\|_q^{(k)} \leqslant [C(q)]^k\, k!\, \|f\|_q; \qquad f \in \mathscr{K}, \tag{0.12}$$

where $C(q)$ only depends on q $(1 < q < +\infty)$.

THEOREM 4. *Let us assume that the semigroup p* $(T_t; t > 0)$ *satisfies the conditions of Theorem* 2 *for some* C_2, *n*, *p*. *Then for every* $0 < \alpha < \beta \leqslant +\infty$, $1 < \beta \leqslant +\infty$, *any* $k = 1, 2, \ldots$, *and any harmonic function* $u(t, x)$ *we have*

$$t^k \left\{ \int_X \left| \frac{\partial^k}{\partial t^k} u(t, x) \right|^\beta dx \right\}^{1/\beta} \leqslant C^k k!\, t^{(n/2)((1/\alpha) - (1/\beta))} M_\alpha(u),$$

where $C = C(C_2, n, p, \alpha, \beta)$.

At this stage I see no joy in running through the rest of the classical Hardy–Littlewood theory (cf. [9, 10]) and generalizing what I can in the setting of a general semigroup. There is one point, however, that I do want to make. This point is related to the theory of classical Lipschitz spaces Λ_α $(0 < \alpha < +\infty)$.

For a general semigroup one is tempted to say that a harmonic function $u(t, x)$ belongs to $\Lambda_\alpha(X)$ for some $0 < \alpha < +\infty$ if

$$\Lambda_\alpha(u; k) = \sup t^{k - \alpha/2} \left\| \frac{\partial^k}{\partial t^k} u(t, \cdot) \right\|_\infty < +\infty; \qquad k > \alpha/2 \tag{0.13}$$

(we use the exponent $\alpha/2$ rather then α because in terms of the classical theory we are using the heat diffusion semigroup rather than the Poisson semigroup). But for the above definition to make sense, the norms $\Lambda_\alpha(u, k)$ $(k = [\alpha/2] + 1,\ [\alpha/2] + 2, \ldots)$ all have to be equivalent. For this to be the case we must impose on T_t additional conditions. One condition that will ensure this is that $(T_t; t > 0)$ is a holomorphic semigroup on $L^1(X)$, i.e., that (0.12) is verified with $p = k = 1$ ($p = 1$ is the "end point" of analyticity, and, in general, analyticity brakes down there).

Analyticity on $L^1(X)$ is a very strong condition indeed. In terms of harmonic analysis on a Lie group G, analyticity on $L^1(G)$ holds for some nilpotent groups (cf. Sect. 4), but very little else! In terms of the spectral analysis of the Laplace–Beltrami operator Δ on a complete non-compact Riemannian manifold M, analyticity of $e^{-t\Delta}$ essentially holds if $\mathrm{Ric}(M) \geqslant 0$ (cf. Sect. 5), but again very little else.

If we do make the assumption of analyticity on $L^1(X)$, however, we again live in the best of all worlds! The spaces Λ_α can then be well defined by any of the norms (0.13) (that are all equialent). Observe also that with the above definition, if we assume in addition that $\dim(T_t) = n$ $(2 \leqslant n < +\infty)$, then $\Lambda_\alpha \subset \Lambda_\beta \subset L^\infty$ $(\alpha > \beta)$ [Indeed, for simplicity, assume that $0<\alpha<2$; then since, by Theorem 1, $u(t,\cdot)\in L^\infty$, we also have $u(0+,\cdot)=u(t,\cdot)-\int_0^t (\partial/\partial s)\,u(s,\cdot)\,ds\in L^\infty$].

If we define just as in the classical case $J_\alpha=(I+A)^{-\alpha/2}$, the Bessel potential, then it is clear that $J_\alpha(\Lambda_\beta)\subset\Lambda_{\alpha+\beta}$ $(\alpha,\beta>0)$. In fact J_α is a Banach space isomorphism between Λ_β and $\Lambda_{\alpha+\beta}$ (indeed $f\in\Lambda_{\alpha+2}\Rightarrow f,\ Af\in\Lambda_\alpha\Rightarrow (I+A)f\in\Lambda_\alpha$).

More general spaces $\Lambda^\alpha_{p,q}$ (of the type considered in [10, 11]) can also be defined, and if $p>1$ we do not even need the analyticity of T_t on $L^1(X)$ to do so. We shall let the reader work out for himself what is true, and what is not, in this general setting. The only point that we shall prove is that the classical duality between H_p and Λ_α with $0<p<1$, $\alpha=n[1/p-1]$, $n=\dim(T_t)$ holds in the above abstract setting, in the sense that

$$|\langle f,\varphi\rangle_X|\leqslant C\,\|f\|_p\,\|\varphi\|_{\Lambda_\alpha};\qquad f\in H_p,\quad \varphi\in\Lambda_\alpha$$

(cf. [16]).

Remark. One way to ensure analyticity of T_t on $L^1(X)$ is to replace T_t by a subordinated semigroup $\hat{T}_t=e^{-tA^\lambda}(0<\lambda<1)$, e.g., the Poisson semigroup $T_t=e^{-tA^{1/2}}$ (cf. Sect. 5). These semigroups are always holmorphic on $L^1(X)$ (cf. [8]).

1. A Technical Point

Let $u(t,x)$ be a nonnegative subharmonic function. By (0.6), for every "test function" $0\leqslant\psi\in\mathscr{K}$ and $t,h>0$, we have

$$\left(\frac{u(t+h,\cdot)-u(t,\cdot)}{h},\psi\right)_{L^2}\leqslant\left(\frac{T_h-I}{h}u(t,\cdot),\psi\right)_{L^2}.$$

Letting $h\to 0$ (and assuming that the corresponding limits exist) we obtain that

$$\left(\frac{\partial u}{\partial t},\psi\right)\leqslant -Q(u,\psi).$$

If we fix now $p\geqslant 2$ and set $v(t,x)=u^{p/2}(t,x)$ we have, at least "formally,"

$$\frac{\partial}{\partial t}\left(\int_X v^2\,d\xi\right)=p\int u^{p-1}\frac{\partial u}{\partial t}\,d\xi\leqslant -pQ(u,u^{p-1})\leqslant -\varepsilon Q(v,v),\qquad\text{(F)}$$

where $\varepsilon=4(1-1/p)$. The last inequality follows from the next lemma.

In this section we shall examine exact conditions under which the above "formal" relation (F) actually holds.

LEMMA. (i) $(x^\alpha - y^\alpha)(x^\beta - y^\beta) \geqslant \alpha\beta(x-y)^2$; $x, y, \alpha, \beta \geqslant 0$, $\alpha+\beta=2$

(ii) $((I-P)\, f^\alpha, f^\beta)_{L^2} \geqslant \alpha\beta((I-P)f, f)$; $\alpha, \beta \geqslant 0$, $\alpha+\beta=2$, $f \in L^2$, $f \geqslant 0$ *and any submarkovian symmetric operator* P *on* L^2.

(iii) *For any Dirichlet form* Q *(say as in* (0, 1)) *any* $0 \leqslant f \in V$ (= *the domain of* Q) *and any* $\alpha, \beta \geqslant 0$ *such that* $f^\alpha, f^\beta \in V$ *and* $\alpha+\beta=2$, *we have*

$$Q(f^\alpha, f^\beta) \geqslant \alpha\beta Q(f, f).$$

Proof. (i) $x^\alpha - y^\alpha = \alpha \int_x^y t^{\alpha-1}\, dt$ and use Hölder.

(ii) Let $K(x, y)$ be the kernel of the operator P(*i.e.*, $Pf(x) = \int_X K(x, y) f(y)\, dy$) we have then

$$((I-P)f, f) = \int_X (1-\sigma(x))\, |f(x)|^2\, dx$$

$$+ \tfrac{1}{2} \int_X \int_X K(x, y)\, |f(x) - f(y)|^2\, dx\, dy,$$

where $0 \leqslant \sigma(x) = \int K(x, y)\, dy = P1(x) \leqslant 1$ (and where $K(x, y)$ in the double integral has to be interpreted as a σ-finite positive measure on $X \times X$ (cf. [1, Chap. III]). (ii) is therefore an immediate consequence of (i).

(iii) This is an immediate consequence of (ii) and (0.1).

Let $\varphi \in \mathscr{K}$ and $\varphi(t, x) = T_t\varphi(x)$, let also $\partial\varphi/\partial t \in L^p(X)$ $(1 < p < +\infty)$ be the L^p-derivative of $\varphi(t, x)$ [in the sense that $\|(1/h)(\varphi(t+h, \cdot) - \varphi(t, \cdot)) - \partial\varphi/\partial t\|_p \to_{h\to 0} 0$; cf. Sect. 0.3]. If we assume that $p \geqslant 2$, $0 \leqslant \varphi \in \mathscr{K}$, then $\varphi(t, \cdot) \in D_A$ $(t > 0)$ (cf. [8]), and by elementary functional analysis it follows that

$$\frac{\partial}{\partial t}\left\{\int_X \varphi^p(t, x)\, dx\right\} = p \int_X \varphi^{p-1} \frac{\partial\varphi}{\partial t}\, dx = -p(\varphi^{p-1}, A\varphi)_{L^2}. \tag{1.1}$$

It is also true that $\varphi^r(t, \cdot) \in V$ $(r \geqslant 1, t > 0)$. Indeed if $r > 1$, $\psi(x) \in V$ and $0 \leqslant \psi(x) \leqslant \frac{1}{2}\min[1, r^{1/1-r}]$, then $\psi^r(x)$ satisfies $|\psi^r(x)| \leqslant |\psi(x)|$ and $|\psi^r(x) - \psi^r(y)| \leqslant |\psi(x) - \psi(y)|$ $(x, y \in X)$, i.e., ψ^r is a "normal construction" of ψ in the sense of [1, Chap. II]. It follows therefore that $\psi^r \in V$, (cf. [1, Chap. V], cf. also our lemma where we have implicitely proved that normal contractions operate on V).

It follows therefore from (1.1) and part (iii) of the lemma that the relation (F) holds with

$$u(t, x) = \varphi(t, x) = T_t\varphi(x); \qquad t > 0, \quad 0 \leqslant \varphi \in \mathscr{K}. \tag{1.2}$$

[Just to make a nervous reader happy, here are the details behind (1.1). We have for $t > 0$

$$\frac{1}{h}\int_X [\varphi^p(t+h, x) - \varphi^p(t, x)]\, dx$$

$$= \int_X p\, \frac{\varphi(t+h, x) - \varphi(t, x)}{h}\, \Theta(t, h, x)\, dx, \qquad (1.3)$$

where Θ is an intermediate value between $\varphi^{p-1}(t+h, x)$ and $\varphi^{p-1}(t, x)$. But because of the elementary inequality $|X^{p-1} - Y^{p-1}|^q \leqslant C(q)\, |X - Y|\, |X + Y|^{p-1}$ $(X \geqslant Y \geqslant 0)$, we have $\varphi^{p-1}(t+h, x) \to_{h \to 0} \varphi^{p-1}(t, x)$ in the norm topology of $L^q(X)$ with $q = p/(p-1)$ and therefore also $\Theta \to_{h \to 0} \varphi^{p-1}(t, x)$. It follows therefore that we can pass to the limit inside the integral of the right-hand side of (1.3) and obtain the middle term of (1.1).]

(F) also holds for the more general choice of u:

$$u(t, x) = |T_t \varphi(x)| = |\varphi(t, x)|; \qquad \varphi \in \mathscr{K}.$$

This, however, is considerably more difficult to prove and will not be needed.

2. The Heart of the Matter: The Moser Iterative Process for Semigroups

Let $(T_t, t > 0)$ be a submarkovian semigroup (as in Sect. (0.1)) such that $\dim(T_t) = n \geqslant 2$ and let $u(t, x)$ be a nonnegative subharmonic function for which the relation (F) of Section 1 is verified. The following estimate holds then. (This in fact is the powerhouse of this paper.)

$$\operatorname*{essup}_{x \in X} u^2(1, x) \leqslant C \int_{1/2}^{2} \int_X u^2(t, x)\, dx\, dt, \qquad (\mathrm{M})$$

where C only depends on n and the C_1 of (0.4). The proof of this inequality is difficult. A detailed account was given in [5, Sect. 6] in a specific case. The proof given there applies in our general context!

Indeed we define as in Section 1 (and as in [5, Sect. 6]) $v(t, x) = u^{p/2}(t, x)$ (for some fixed $p \geqslant 2$) and we let $\psi(t) \geqslant 0$ be some smooth function for $t \geqslant 0$ to be specified later. We can then calculate

$$\frac{\partial}{\partial t}\left\{\int_X \psi^2 v^2\, d\xi\right\} = \psi^2 \frac{\partial}{\partial t}\left\{\int_X v^2\, d\xi\right\} + 2\psi\psi_t \int_X v^2\, d\xi.$$

A use of (F) allows us to conclude that

$$\frac{\partial}{\partial t}\left\{\int_X \psi^2 v^2\, d\xi\right\} + \varepsilon\psi^2 Q(v, v) \leqslant 2\int_X |\psi\psi_t v^2|\, d\xi$$

with $\varepsilon = 4(1-1/p)$.

From there onwards the proof proceeds "verbatum" and with the same notations as in [5, Sect. 6]; it would therefore be a waste of good paper to repeat it.

Observe, however, that in [5, Sect. 6] we have, in fact, proved (without explicitly saying so) the following sharper inequality:

$$\operatorname*{essup}_{x\in X} u^2(1, x) \leqslant C\int_{1/2}^{1}\int_X u^2(t, x)\, dt\, dx. \tag{M$'$}$$

The weaker inequality (M) suffices for all the applications, but one should note that (M′) is more natural since it only involves the "*parabolic nhb*" of the line $t=1$ in $(0, +\infty)\times X$ (and of course both (M) and (M′) are none other than the "Moser parabolic estimates" of [13, 14], adapted to abstract semigroups).

The way one uses inequality (M) to prove that $(A)\Rightarrow(B)$ in Theorem 1 was also explained (in a disguised form) in [5] (end of Sect. 5). For clarity, however, we shall repeat the argument.

Fix some $\sigma>0$ and consider

$$\tilde{T}_t = T_{t/\sigma^2};\quad \tilde{\xi} = \sigma^n\xi;\quad \tilde{Q} = \sigma^{n-2}Q;\quad \tilde{u}(t, x) = u(t/\sigma^2, x). \tag{2.1}$$

It is clear that $\tilde{T}_t$ is a submarkovian semigroup on $(X, \tilde{\xi})$ and that $\dim(\tilde{T}_t) = n$ uniformly in σ (i.e., the constant C_1 that appears in (0.4) is independent of σ). Furthermore $\tilde{u}$ is subharmonic for the above semigroup. $\tilde{u}$ satisfies therefore the estimate (M) uniformly in σ (i.e., C in (M) does not depend on σ).

If we substitute and compute we obtain

$$\operatorname*{essup}_{x\in X} u^2(t, x) \leqslant Ct^{-(n+2)/2}\int_{t/2}^{2t}\int_X u^2(t, x)\, dt\, dx \tag{M$_t$}$$

and this implies that

$$\|u(t, x)\|_\infty \leqslant Ct^{-n/4}\sup_{s>0}\|u(s, x)\|_2. \tag{2.2}$$

Now, as we have shown in Section 1 the above estimate can be applied to the following choice of u:

$$u(t, x) = T_t\varphi(x),\qquad 0\leqslant\varphi\in\mathscr{K}.$$

We see therefore that the estimate (2.2) implies that T_t satisfies the conditions of Theorem 2 (with $p=2$). The conclusion is that the condition (B) of Theorem 1 holds.

An alternative way of showing how from (M_t) one can deduce that $(A) \Rightarrow (B)$ (in Theorem 1) was explained in [7].

For one of our applications it will be important to have the exact value of the constant C in (M) (or (M′)) in terms of the dimension n and the C_1 of (0.4).

In fact we can take

$$C = \varphi(n)\, C_1^n = \exp(\tfrac{1}{2} n \log n + An)\, C_1^n, \tag{2.3}$$

where A is a numerical constant.

The dependence C_1^n is of course no surprise and can be seen directly by rescaling. Indeed, if we replace ξ by $\xi_\lambda = \lambda\xi$ and Q by $Q_\lambda = \lambda Q$ (and leave $V_\lambda = \operatorname{dom}(Q_\lambda) = V$) we see that the semigroup T_t does not change and therefore C_1 in (0.4) is replaced by $\lambda^{-1/n} C_1$. If we chose $\lambda = C_1^n$ we obtain then (M) with a constant C that only depends on n but with ξ replaced by ξ_λ. This shows that $C = \varphi(n)\, C_1^n$.

To obtain the exact form of $\varphi(n)$ one has to go through the whole proof of Section 6, [5] and make a slight improvement at the end. In that proof immediately after the relation (6.5) we had chosen $\rho_\nu = 10^{-10} 2^{-\nu}$ $(\nu = 1, 2, \ldots)$. We shall now choose them differently and set

$$\rho_0 = 0, \quad \rho_\nu = \frac{10^{-10}}{n} \kappa^{-\nu}; \qquad \nu \geqslant 1, \quad \kappa = 1 + \frac{2}{n}.$$

The point of this choice is that it optimises $\mathscr{E} = -\sum_j \log \rho_j/\kappa^j$ under $\sum_j \rho_j = \frac{1}{2}\, 10^{-10}$. It gives in fact $\mathscr{E} = \frac{1}{2} n \log n + O(n)$. If we finish up the proof of Section 6, [5] with this choice of ρ_ν's we obtain the required value of $\varphi(n)$ as in (2.3).

From (2.3) it follows as before that (2.2) can be sharpened and we have (under condition (0.4)) the following estimate:

$$\|T_t f\|_\infty \leqslant \varphi(n)^{1/p}\, C_1^{n/p}\, \|f\|_p; \qquad f \in L^p(X), \quad 1 \leqslant p \leqslant +\infty. \tag{2.4}$$

To see (2.4) for $p=1$ we can argue as [7], all the other values of p follow by interpolation.

3. The Hardy–Littlewood Theory

(3.1) *Proof of Theorem* 2

It is clearly enough to prove Theorem 2 for $0<\alpha<p$ and $\beta=+\infty$ (the rest of the range $0<\alpha<\beta\leqslant+\infty$ follows by routine interpolation). The proof depends on a device that has its origin in [15] and which consists of first proving the following:

Lemma. *Let* $(T_t;t>0)$ *satisfy the conditions of Theorem* 2 *let* $0<\alpha<p$ *and let* $u(t,x)$ *be a subharmonic function with respect to* T_t *and let* $m_\alpha(t)$ *be as in* (0.7). *Let us further assume that*

$$m_\infty(t)<+\infty;\qquad t>0. \tag{*}$$

We have then

$$m_\infty^\alpha(2^k)\leqslant C_3\int_1^2 m_\alpha^\alpha(t)\,dt, \tag{3.1}$$

where $k=(2p-\alpha)/2(p-\alpha)$ *and* C_3 *only depends on the* C_2, n, p *of* (0.8) *and on* α.

Proof. Assume that

$$m_\infty(t)\geqslant 1,\qquad 0<t\leqslant 2^k;\qquad \int_1^2 m_\alpha^\alpha(t)=1$$

for otherwise there is nothing to prove.

Standard convexity and our hypothesis implies that for $0<t<s$ we have

$$\begin{aligned}\log m_\infty(s)&\leqslant\log[C(s-t)^{-n/2p}]+\log m_p(t)\\&\leqslant\log[C(s-t)^{-n/2p}]+\left(1-\frac{\alpha}{p}\right)\log m_\infty(t)+\frac{\alpha}{p}\log m_\alpha(t).\end{aligned}$$

Set $s=t^k\geqslant t$ and integrate the above inequality between 1 on 2 with respect to dt/t,

$$\begin{aligned}\frac{1}{k}\int_1^{2^k}\log m_\infty(t)\frac{dt}{t}\leqslant C+\left(1-\frac{\alpha}{p}\right)\int_1^2\log m_\infty(t)\frac{dt}{t}\\+\frac{\alpha}{p}\int_1^2\log m_\alpha(t)\frac{dt}{t}.\end{aligned}$$

Observe that

(i) $k > 1$ and $\log m_\infty(t) > 0$, $0 < t \leqslant 2^k$.

(ii) $(1/t) \log m_\alpha(t) \leqslant C m_\alpha^\alpha(t)$, $1 \leqslant t \leqslant 2$.

The conclusion is that

$$0 \leqslant \left(\frac{1}{k} - \frac{p-\alpha}{p}\right) \int_1^2 \log m_\infty(t) \frac{dt}{t} \leqslant C;$$

and therefore

$$\log m_\infty(2^k) \leqslant \log m_\infty(t_0) \leqslant C;$$

for some $1 \leqslant t_0 \leqslant 2$ [All the C's that appear in the above proof are not necessarily the same but they all only depend on C_2, n, p, and α]. This ends the proof of the lemma.

The proof of Theorem 2 follows by renormalization. Fix $\sigma > 0$ and define $\tilde{T}_t$, $\tilde{\xi}$, $\tilde{u}$ as in (2.1). The conclusion of the lemma holds for $\tilde{u}$, $\tilde{\xi}$ uniformly in σ [i.e., in (3.1) C_3 is independent of σ] because the hypothesis (0.8) holds uniformly in σ. Upon writing the estimate (3.1) down for $\tilde{u}$, $\tilde{\xi}$ we see that Theorem 2 follows with $0 < \alpha < p < \beta = +\infty$; provided that the condition $(*)$ of the lemma is verified. It is easy to eliminate that additional hypothesis:

(i) If $1 \leqslant \alpha < +\infty$ to eliminate $(*)$ observe that $L^\infty \cap L^\alpha$ is dense in L^α and we can use the standard "a priori estimate" argument.

(ii) If $0 < \alpha < 1$ let us fix some level $t = t_0 > 0$ where we have (by (0.5)) $u(t_0, \cdot) \in L^1 + L^\infty$. But then the previous case (i) shows that $u(2t_0, \cdot) \in L^\infty$. $t_0 > 0$ is arbitrary and we are done.

(3.2) *Proof of Theorem* 4

If $\alpha > 1$ it suffices to use the estimate (0.12) with $q = \alpha$ and then apply Theorem 2 to the harminic function $U(t, x) = (\partial^k/\partial t^k)\, u(t, x)$.

To prove the case $0 < \alpha \leqslant 1 < \beta \leqslant +\infty$ of Theorem 3 we first estimate $m_q(t/2)$ for some $1 < q < \beta$ (with the use of Theorem 2). Then we use the previous case with $\alpha = q$ and the semigroup property to estimate $\|(\partial^k/\partial t^k)\, u(t, x)\|_\beta$. This completes the proof.

(3.3) *Proof of Theorem* 3

For every $u \in H_\alpha$ and any t, $T > 0$, $z \in \mathbb{C}$, $0 > \operatorname{Re} z = \gamma > -n$ and $x \in X$ we have

$$G_z u(t, x) = \int_0^T s^{z/2-1} u(t+s, x)\, ds + \int_T^\infty s^{z/2-1} u(t+s, x)\, ds.$$

It follows therefore that

$$|G_z u(t,x)| \leqslant \frac{2}{\delta} T^{\gamma/2} u^*(x) + C \|u\|_\alpha \int_T^\infty s^{(\gamma/2)-1}(t+s)^{n/2\alpha}\, ds,$$

where C only depends on n on α and the C_1 of (0.4). The conclusion is that

$$|G_z u^*(x)| \leqslant \frac{2}{\gamma} T^{\gamma/2} u^*(x) + C T^{(\gamma/2)-(n/2\alpha)} \|u\|_\alpha$$

if we optimize the above estimate by choosing $T^{-n/2\alpha} = (u^*(x)/\|u\|_\alpha)$ we obtain

$$|G_z u^*(x)| \leqslant C(u^*(x))^{\alpha/\beta} \|u\|_\alpha^{1-(\alpha/\beta)}.$$

The theorem follows.

(3.4) *Proof that* (B) $\Rightarrow$ (A) *in Theorem* 1. This is an immediate consequence of Theorem 3 and (0.11) for indeed condition (A) simply says that

$$\|f\|_{2n/(n-2)} \leqslant C_1 \|A^{1/2} f\|_2, \qquad \forall f \in \mathscr{K} \cap D_{A^{1/2}}.$$

(3.5). *The* $(H_p; \Lambda_\alpha)$ *Duality*: Let $f, \varphi \in \mathscr{K}$ and let $f(t,x) = T_t f(x)$ and $\varphi(t,x) = T_t \varphi(x)$. By elementary spectral Theory we have (cf. [3])

$$\langle f, \varphi \rangle_X = C(k) \iint t^{2k-1} \frac{\partial^k}{\partial t^k} f(t,x) \frac{\partial^k}{\partial t^k} \varphi(t,x)\, dt\, dx; \quad k = 1, 2, \ldots.$$

Now let $\alpha > 0$ and let $k > \alpha/2$. It follows that

$$\begin{aligned} |\langle f, \varphi \rangle_X| &\leqslant C(k) \|f\|_{\Lambda_\alpha} \int_0^\infty t^{k+(\alpha/2)-1} \left\| \frac{\partial^k}{\partial t^k} \varphi(t,\cdot) \right\|_1 dt \\ &\leqslant C \|f\|_{\Lambda_\alpha} \int_0^\infty t^{(\alpha/2)-1} \|\varphi(t/2,\cdot)\|_1\, dt \\ &= C \|f\|_{\Lambda_\alpha} \int_0^\infty \int_X t^{(\alpha/2)-1} |\varphi(t,x)|\, dt\, dx. \end{aligned}$$

The last estimate follows from the hypothesis that the semigroup T_t is holomorphic on $L^1(X)$. But the integral $\Phi(x) = \int_0^\infty t^{(\alpha/2)-1} |\varphi(t,x)|\, dt$ can be estimated exactly as in the proof of Theorem 3 and we obtain that $\Phi(x) \leqslant C\varphi^*(x)^p. \|\varphi\|_p^{1-p}$ provided that $0 < p < 1$ and $1 = (1/p) - (\alpha/n)$. The result follows.

4. An Example: The Dilation Structure—Nilpotent Groups

Let us assume that (X, ξ) $(T_t; t>0)$, V, Q, and A are as in Section (0.1) and, for simplicity, let us assume here that X is a locally compact, countable at infinity space and ther ξ is a Radon measure on X. For every $r>0$ let $\delta_r: X \to X$ be a homeomorphism of X such that $\delta_r \circ \delta_s = \delta_{rs}$, $\delta_1 = Id$ and such that $\delta_r \to_{r \to 1} Id$ in the uniform on compact a topology. We shall say that $(\delta_r; r>0)$ is a dilation structure for our Dirichlet space (or simply for the semigroup (T_t)) if $\delta_r(V) = V$, $\delta_r(D_A) = D_A$ $(r>0)$ and if

$$\int_X (f \circ \delta_r)\, d\xi = r^{-d} \int_X f\, d\xi; \qquad \forall f \in \mathscr{K};$$

$$Q(f \circ \delta_r, g \circ \delta_r) = r^{-d+2} Q(f, g), \qquad \forall f, g \in V \tag{4.1}$$

for some fix $d \in \mathbb{R}$. [Example: $(X, \xi) = (\mathbb{R}^d, Leb)$, $Q(f, f) = \int_{\mathbb{R}^d} |\nabla f|^2\, dx$ and $\delta_r(x) = r \cdot x$ $(x \in \mathbb{R}^d)$]. Let us also denote by $\check{\delta}_r(f)(x) = f \circ \delta_r(x)$ $(f \in \mathscr{K}$, $\check{\delta}_r(f) \in \mathscr{K}$, $x \in X)$.

For every fixed $t, r>0$ it is clear then that $f \mapsto \tilde{T}_t f = (\check{\delta}_r \circ T_{r^2 t} \circ \check{\delta}_{r^{-1}})(f)$ is a submarkovian symmetric operator and that $(\tilde{T}_t; t>0)$ is a symmetric submarkovian semigroup. One easily verifies from (4.1) that the Dirichlet form $\tilde{Q}$ attached to $\tilde{T}_t$ is the same as that of T_t, i.e., $\tilde{Q} = Q$ with the same $\mathrm{Dom}(\tilde{Q}) = V$. The conclusion is that

$$T_t = \check{\delta}_r \circ T_{r^2 t} \circ \check{\delta}_{r^{-1}}; \qquad t, r>0 \tag{4.2}$$

and therefore also

$$T_t = \check{\delta}_{1/\sqrt{t}} \circ T_1 \circ \check{\delta}_{\sqrt{t}}. \tag{4.3}$$

If we differentiate (4.2) with respect to t (for fixed r), we obtain

$$t \frac{\partial}{\partial t} T_t = t r^2 \check{\delta}_r \circ \left(\frac{\partial}{\partial s} T_s \Big|_{s = r^2 t} \right) \circ \check{\delta}_{r^{-1}}.$$

If in the above relation we set $s = tr^2 = 1$ we obtain

$$t \frac{\partial}{\partial t} T_t = \check{\delta}_{1/\sqrt{t}} \circ \frac{\partial}{\partial s} T_s |_{s=1} \circ \check{\delta}_{\sqrt{t}}, \qquad t>0 \tag{4.4}$$

Under the additional hypothesis

$$\frac{\partial T_s}{\partial s} f |_{s=1} \in L^1, \qquad T_1 f \in L^\infty; \quad \forall f \in L^1(X) \tag{H}$$

we deduce from (4.3) and (4.4) that T_t is holomorphic on $L^1(X)$ and that part (B) of Theorem 1 is verified with $n=d$; so, if $d>2$, we have

$$\dim(T_t)=d.$$

When X is a C^∞ manifold and A is a differential operator of the 2nd order then the hypothesis (H) is connected with the hypoellipticity of A.

The above situation arises, for instance, in the theory of homogeneous groups (i.e., Lie groups with a stratified Lie algebra (cf. [17, p. 5] for a definition). If G is such a group we can set $A=-(X_1^2+\cdots+X_p^2)$ where $X_1,\dots,X_p$ span the top layer of the stratification of the Lie algebra of G.

The hypothesis (H) is then certainly verified. It follows in particular that $\|e^{-tB}f\|_\infty \leqslant Ct^{-d/2}\|f\|_1$, where d is the homogeneous dimension of G and B is *any* invariant elliptic operator on G.

We shall not elaborate on this example because the literature on stratified Lie groups and their dilation structure is already quite exhaustive (-ing)! (e.g., [17, 18]). We simply wanted to connect our definition of dimension with ideas that are already well known.

In [29] [30] I consider the same problem for a *general nilpotent* group.

5. The Poisson Semigroup and Geometric Applications

Given any submarkovian symmetric semigroup $T_t=e^{-tA}$ it is well known that we can associate, by the principle of subordination (cf. [3; 8, Chap. IX]), the corresponding Poisson semigroup

$$P_t=e^{-tA^{1/2}}=\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\,T_{t^2/4u}\,du.$$

It is then an immediate consequence of the equivalence (A)$\Leftrightarrow$(B) in Theorem 1 and of Theorem 3 that for all $n>2$ we have

$$\dim(T_t)=n \Leftrightarrow \dim(P_t)=2n.$$

An analogous fact clearly also holds for every A^λ, $0<\lambda<1$, and the corresponding subordinated semigroup (cf. [8]).

Let now M be complete connected Riemannian manifold such that

$$\operatorname{Ric}(M)\geqslant 0; \qquad \dim(M)=n>2$$

and let $B_t(m)=\{m'\in M,\ \operatorname{dist}(m',m)\leqslant t\}$ $(m\in M)$. By standard Differential Geometry (cf. [19]) it follows that

$$\operatorname{Vol} B_t(m)\leqslant k(n)\,t^n; \qquad t>0, \quad m\in M,$$

where $k(n)$ denotes the euclidean volume of the unit ball in $\mathbb{R}^n$. What is also true (cf. [20](is that there exists $C = C(n)$ such that the Poisson Kernel [defined by $P_t f(m') = \int_M K_t(m', m) f(m)\, d\,\mathrm{Vol}$] satisfies

$$\frac{1}{C \operatorname{Vol} B_t(m)} \leqslant K_t(m, m) \leqslant \frac{C}{\operatorname{Vol} B_t(m)}; \qquad t > 0, \quad m \in M.$$

From Theorem 1 and the above considerations, we deduce at once that for a Riemannian manifold as above the following two conditions are equivalent:

(I) There exists $c > 0$ s.t.

$$\operatorname{Vol} B_t(m) \geqslant ct^n; \qquad t > 0, \quad m \in M.$$

(II) There exists $c_2 > 0$ s.t.

$$\left(\int_M |f|^{2n/(n-2)}\, d\,\mathrm{Vol}\right) \leqslant c_2 \left(\int_M |\nabla f|^2\, d\,\mathrm{Vol}\right)^{1/2}; \qquad f \in C_0^\infty(M).$$

The above condition (II) is closely related to the classical "isoperimetric inequality":

$$\left(\int_M |f|^{n/(n-1)}\, d\,\mathrm{Vol}\right)^{(n-1)/n} \leqslant c_1 \int_M |\nabla f|\, d\,\mathrm{Vol}; \qquad f \in C_0^\infty(M) \qquad \text{(Iso)}$$

with c_1 independent of f. In fact it is well known, and easy to prove (cf. [21, 22]) that (Iso) $\Rightarrow$ (II). It is just as easy (even easier in fact) to see directly that (Iso) $\Rightarrow$ (I).

What is also true is that for M as above we also have (Iso) $\Leftrightarrow$ (I) (in fact, this true for all $n \geqslant 1$ and not only for $n > 2$). The proof, however, requires new ingredients and we shall not give it here. We shall come back to that in a later paper.

A Final Observation

For M as above not only P_t is holomorphic on $L^1(M; d\,\mathrm{Vol})$ but we also have the following pointwise estimate:

$$t \left| \frac{\partial}{\partial t} P_t f(m) \right| \leqslant C P_t f(m); \qquad 0 \leqslant f \in C_0^\infty(M),$$

where C is independent of f (cf. [20]).

If injectivity rad. $\geqslant \delta > 0$ then the heat diffusion semigroup $e^{-t\Delta}$ is also holomorphic on $L^1(X)$. The proof of that fact relies on the standard volume growth estimates under $\mathrm{Ric}(M) \geqslant 0$ (cf. [19]) and on the following deep estimate for the heat diffusion kernel $p_t(x, y)$

$$\left|\frac{\partial}{\partial t} p_t(x, y)\right| \leqslant C t^{-(n/2)-1} \exp\left(-\frac{d^2(x, y)}{ct}\right); \qquad x, y \in M, \quad 0 < t < 1$$

(cf. [12, 27]).

In fact the above shows that $e^{-t\Delta}$ is holomorphic as soon as

$$\mathrm{Ric}(M) \geqslant -K; \qquad \mathrm{Vol}\, B_t(m) \leqslant C t^n,$$

where C is independent of m and t ($n =$ topological dimension of M).

6. The Orlicz Spaces and Soluble Groups

In this Section I shall consider a semigroup $(T_t; t > 0)$ as in Section 0.1 that satisfies

$$\|T_t f\|_\infty \leqslant C \exp(-ct^{1/m}) \|f\|_1; \qquad f \in L^1(X), \qquad t > 0. \tag{6.1}$$

for some fixed C, c, $m > 0$. A number of comments are in order:

(i) If we let $t \to 0$ in (6.1) we easily see that $\|f\|_2 \leqslant C \|f\|_1$ this forces the space (X, ξ) to be discrete. We may therefore, and we shall, assume in what follows $\xi(\{x\}) = 1$, $\forall x \in X$ (and X to be contable).

(ii) It is well known, and easily proved by elementary spectral theory that (6.1) with $m = 1$ is equivalent to

$$\|f\|_2 \leqslant C Q^{1/2}(f, f); \qquad f \in L^2(X) \tag{6.2}$$

with C independent of f [The way to see it, is to prove that both (6.1) with $m = 1$ and (6.2) are equivalent to

$$\inf\{t/t \in \text{spectrum of } A\} > 0].$$

(iii) Let G be a finitely generated soluble group that is not almost nilpotent (i.e., that is not a finite extension of a nilpotent group). We showed then in [24] that it is possible to chose $\mu \in \mathbb{P}(G)$ a symmetric probability measure with finite support that satisfies

$$\mu^n(\{e\}) = O(\exp(-cn^{1/3}))$$

for some $c > 0$ [where $e \in G$ is the neutral element of G]. If we let then $T_t = \exp[t(\mu - \delta_e)]$ be the corresponding convolution semigroup it is clear that T_t satisfies (6.1) with $m=3$. Let us now introduce some notations: For $f \in \mathscr{K}$ let $f^*(x) = \sup_{t>0} |T_t f(x)|$.

For every $m>0$ let

$$\Phi(t) = \Phi_m(t) = \begin{cases} t^2(-\log t)^{-m}, & 0<t<\frac{1}{2}, \\ At^2+B, & t \geqslant 2 \quad (A=A(m), \quad B=B(m)) \end{cases}$$

with $\Phi(0)=0$ and the $A, B \in \mathbb{R}$ chosen so that $\Phi(t)$ can be extended to a continuous convex function of $t \geqslant 0$. Let us also denote by $\| \ \|_{\Phi_m}$ the Orlicz norm induced by the above function (cf. [25, 26]).

By the elementary inequality:

$$x^p \leqslant e^m \left(\frac{m}{p-2}\right)^m x^2(-\log x)^{-m}; \qquad 0<x<1, \quad m>0, \quad p>2,$$

it follows that

$$\|F\|_p \leqslant C(m) \left(\frac{1}{p-2}\right)^{m/2} \|F\|_{\Phi_m}; \qquad F \in \mathscr{K}, \quad 2<p<+\infty, \tag{6.3}$$

where $C(m)$ only depends on m (observe that $\|F\|_{\Phi_m} \geqslant C_1(m) \|F\|_\infty$).

We have then

PROPOSITION. *There exists $K=K(C, c, m)>0$ depending only on C, c, and m such that for every semigroup $(T_t; t>0)$ that satisfies* (6.1) *and every $f \in L^2(X)$, $\|f\|_2 \leqslant 1/K$, we have*

$$\|G_1 f(x)\| \leqslant Kf^*(x)[-\log f^*(x)]^{m/2} \qquad (x \in X),$$

where G_1 is as in (0.10).

The proof follows the same lines as the proof of Theorem 3 except that we must now choose the optimizing parameter $T=[-k \log f^*(x)]^m$ with an appropriate choice of $k>0$ (observe that then by our hypothesis $T \geqslant (k \log K)^m$).

From the above proposition and the fact that $\|f^*\|_2 \leqslant A \|f\|_2$ ($f \in \mathscr{K}$ and A numerical, cf. [3]) it follows at once that for T_t as in the proposition we have:

$$\|G_1 f\|_{\Phi_m} \leqslant K_1 \|f\|_2; \qquad f \in \mathscr{K} \tag{6.4}$$

or, equivalently:

$$\|f\|_{\Phi_m} \leqslant K_1 Q^{1/2}(f, f); \qquad f \in \mathscr{K} \cap V. \tag{6.4'}$$

Equation (6.4′) combined with (6.3) gives

$$\|f\|_{2n/(n-2)} \leqslant K_2 n^{m/2} Q^{1/2}(f,f); \qquad f \in \mathscr{K} \cap V, \quad 2 < n < +\infty. \tag{6.5}$$

K_1, K_2 in the above inequalities are two constants that depend on C, c, and m.

The above inequalities are *not* sharp as far as m is concerned. Indeed for $m=1$ (6.2) is valid and this is sharper them (6.4′) with $m=1$. Equation (6.4′), however, *has* a *sharp* converse. Indeed we have:

PROPOSITION. *Let $(T_t; t>0)$ be as in Section 0.1 (with $\xi(\{x\})=1$, $x \in X$) and let us assume that it satisfies the dimensional inequality (6.4′) for some fixed $K_1, m>0$ (and Q as in (0.1)). Then the semigroup $(T_t; t>0)$ satisfies*

$$\|T_t f\|_\infty \leqslant C \exp(-ct^{1/(m+1)}) \|f\|_1; \qquad \forall f \in L^1,$$

where C and c only depend on K_1 and m.

Indeed the estimates (6.5) (2.4) show that we have

$$\|T_t f\|_\infty \leqslant \exp\left((1+m)\frac{n}{2}\log n + An - \frac{n}{2}\log t\right) \|f\|_1; \qquad f \in L^1$$

valid for all $t>0$ and $n>2$ with an A only depending on K_1 and m. It suffices to optimise by setting $n = \alpha t^{1/(1+m)}$ (for an appropriate $\alpha>0$) to get the proposition.

The sharpness of the above proposition can be seen by letting $m \to 0$ and comparing it with (6.2).

An immediate corollary of the above two propositions is the following.

THEOREM. *Let G be a finitely generated soluble group that is not almost nilpotent. Every symmetric $\mu \in \mathbb{P}(G)$ such that $Gp(\operatorname{supp}\mu)=G$ satisfies then:*

$$\mu^n(\{e\}) = O(\exp(-cn^{1/4}))$$

for some $c>0$ (that depends on μ).

We shall let the reader ponder over the proof (with the help of [5]). Observe that clearly (cf. [24]) the above theorem is not sharp; $\frac{1}{3}$ is no doubt the correct exponent but this is another story!

7. A FINAL REMARK

One draw back of the definition that we gave for the dimension in Section 0.1 is that it does not make much sense in compact situations [e.g., if we consider the standard Laplacian Δ on the 1-dimensional torus $(\mathbb{T}; d\theta)$ then the semigroup $e^{-t\Delta}$ does not satisfy (0.4) for any $n \geqslant 2$]. There is a way to rectify this (at last partially). Indeed the same proofs that we gave in text will also prove the following:

THEOREM. *Let T_t $(t>0)$ be a semigroup as in Section* 0.1 *let Q be as in* (0.1) *and let as assume that $n>2$; then we have*:

$$\|f\|_{2n/(n-2)} \leqslant C_1[Q^{1/2}(f,f)+\|f\|_2]; \qquad \forall f \in \mathscr{K} \cap V, \tag{7.1}$$

(*where $C_1>0$ is independent of f*) *if and only if there exists $C_2>0$ s.t.*

$$\|T_t f\|_\infty \leqslant C_2 t^{-n/2} \|f\|_1; \qquad f \in L^1(X); \quad 0<t<1. \tag{7.2}$$

The point of course is that estimate (M) of Section 2 still holds under our more general condition (7.1). The only thing that changes is the rescaling procedure (2.1). Indeed (7.1) is stable under the rescaling (2.1) only if $\sigma > 1$ and that is why (7.2) only holds for $0<t<1$! ((7.1) $\Rightarrow$ (7.2) even when $n=2$)

The proof the other way around is easy. Indeed for $f \in L^2$ we have

$$\begin{aligned}(1+A)^{-1/2} f(x) &= C \int_0^\infty t^{-1/2} e^{-t} T_t f(x)\,dt \\ &\leqslant C\left[T^{1/2} f^*(x) + \|f\|_2 \int_T^\infty t^{-1/2} e^{-t} \varphi_n(t)\,dt\right] \\ &\leqslant C[T^{1/2} f^*(x) + \|f\|_2\, T^{1/2-n/4}]\end{aligned}$$

with

$$\varphi_n(t) = \begin{cases} t^{-n/4}, & 0<t<1, \\ 1, & t \geqslant 1. \end{cases}$$

We finish the proof as before.

Note added in proof. In [31] I have obtained a generalisation of the above theory in non-symmetric situations, e.g., complex Schrödinger perturbations of Δ. In [30] problems closely related to this paper are considered in the setting of classical Schrödinger semigroups on $\mathbb{R}^n$.

REFERENCES

1. J. DENY, "Potential Theory," C.I.M.E., Ier cycle, Stresa, July 2–10, 1969.
2. M. FUKUSHIMA, "Dirichlet Forms and Markov Processes," North-Holland/Kodansha, Amsterdam, 1980.

3. E. M. STEIN, "Topics in Harmonic Analysis Related to Littlewood–Paley Theory," Princeton Univ. Press, Princeton, N.J., 1970.
4. N. TH. VAROPOULUS, *C. R. Acad. Sci. Ser. I Math.* **298** (1984), 465–468.
5. N. TH. VAROPOULOS, Isoperimetric inequalities and Markov chains, *J. Func. Analysis* **63** (1985), 215–239.
6. N. TH. VAROPOULOS, *C. R. Acad. Sci. Ser. I Math.* **298** (1984), 233–236.
7. N. TH. VAROPOULOS, *C. R. Acad. Sci. Ser. I Math.* **299** (1984), 651–654.
8. K. YOSIDA, "Functional Analysis," Springer-Verlag, Berlin/New York, 1978.
9. P. L. DUREN, "Theory of H^p spaces," Academic Press, New York, 1970.
10. E. M. STEIN, "Singular Integrals and Differentiability Properties of Functions," Princeton Univ. Press, Princeton, N.J., 1971.
11. C. S. HERZ, *J. Math. Mech.* **18** No. (4) (1968), 283–324.
12. S. Y. CHENG, P. LI, AND S. T. YAY. *Amer. J. Math.* **103**, No. 5, 1021–1063.
13. J. MOSER, *Comm. Pure Appl. Math.* **XVII** (1964).
14. J. MOSER, *Comm. Pure Appl. Math.* **XXIV** (1971).
15. G. H. HARDY AND J. E. LITTLEWOOD, *J. Reine Angew. Math.* **167** (1932), 405–423.
16. P. L. DUREN, B. W. ROMBERG, AND A. L. SHIELDS, *Reine Angew. Math.* **238** (1969), 32–60.
17. G. B. FOLLAND AND E. M. STEIN, "Hardy Spaces on Homogeneous Groups," Mathematical Notes, Princeton Univ. Press, Princeton, N.J., 1982.
18. R. W. GOODMAN, "Nilpotent Lie Groups," Lecture Notes in Math., Vol. 562, Springer-Verlag, Berlin/New York.
19. R. L. BISHOP AND R. J. CRITTENDEN, "Geometry on Manifolds," Academic Press, New York, 1964.
20. N. TH. VAROPOULOS, *J. Funct. Anal.* **44** (1981), 359–380.
21. R. OSSERMAN, *Bull. Amer. Math. Soc.* **84**, No. 6 (1978), 1192–1197.
22. S.-T. YAU, *Ann. Sci. Ecole Norm. Sup.* (4) **8** (1975), 487–507.
23. N. LOHOUE, *C. R. Acad. Sci. Ser A* **290** (1980), 605–608.
24. N. TH. VAROPOULOS, A potential theoretic property of soluble groups, *Bull. Sci. Math.* **108**, No. 3 (1984), 263–273.
25. A. ZYGMUND, "Trigonometric Series," Vol. I and II, Cambridge Univ. Press, London, 1959.
26. J. NEVEU, "Martingales a temps discret," Masson & Cie, Paris, 1972.
27. J. CHEEGER, M. GROMOV, AND M. TAYLOR, *J. Differential Geometry* **17** (1982), 17–53.
28. N. TH. VAROPOULOS, *C. R. Acad. Sci. Ser. I Math.* (1985).
29. N. TH. VAROPOULOS, Analysis on nilpotent groups, preprint.
30. E. B. DAVIES AND B. SIMON, *J. Funct. Anal.* **59**, No. 2 (1985), 335–395.
31. N. TH. VAROPOULOS, *C. R. Acad. Sci. Ser. I Math.* **300** (1985), 617–620.

Printed in Belgium